"linear geometry"

Part 1 General Facts abt geometry:
dimension

- line: 1-D
- plane, $\mathbb{R}^2$: 2-D
- space, $\mathbb{R}^3$: 3-D
- point: 0-D

Q: What does it mean to be d-dim?
A: To be parametrized by d independent parameters

not req. to satisfy any eq

e.g. line paramet. by 1 indep. (usually called t) is parametrized by 2 parameters (usually called t, s or t_1, t_2)

line might be given by
$(x, y, z) = (3 + t, 1 + 2t, 1 - 4t)$
↳ vector lurking $(3, 1, 1)$ and $(1, 2, -4)$

e.g. plane might be given by
$(x, y, z) = (3 + t - s, 1 - t + 3s, 4 - 3s)$

Vectors: $\underbrace{(3, 1, 4)}_{\text{const.}}$ and $\underbrace{(1, -1, 0)}_{t}$ and $\underbrace{(1, 3, -3)}_{s}$

Principle: more parameters = higher dim
e.g. $\emptyset$-parameters, e.g. $(x, y, z) = (2, 7, -4)$
↑ a point!

e.g. 3-parameters
$(x, y, z) = (2 + t_1 - t_2, 3 + t_3, 1 - 4t_1)$

this dermines all of space

Can also define geometric object as solution set to an eq.

Principle: more eq. = smaller dim

- Line given by 2 eq.
- Plane given by 1 eq.

e.g. $3x-2y+z=7$ defines a plane

eg $(2,0,1)$ is a point on it

eg. $\left.\begin{array}{l} 3x-2y+z=7 \\ x-5y-3z=4 \end{array}\right\}$

→ common soln set to these eq is a line
⇒ this line lies on that plane

Notice these are linear eqns

Consider non-linear eqns:

$$x^2-3y^2-4z^2=2$$

or $$y^2-x^3-x=7$$

these give surfaces, so 2-dim, but not a plane

These eqns are algebraic
⇒ algebraic geometry

Also use differentiable fcns like

$$\sin^2 x - e^z + y = 2$$

⇒ differential geometry

Caveat usually 2 linear eqs determine a line but, consider the pair of eqs

$$\left.\begin{array}{l} 3x-2y+z=7 \\ 4y-2z-6x=-14 \end{array}\right\}$$

→ This determines a plane, not a line

More precise principle
more eqns that are independent from each other means lower dimension.

eg 3 independent eqs, in 3 vars determines a point

eg 3 independent eqs, in 3 vars determines
a point

Part 2 Lines & Parametric Eqs

Recall $(x,y,z) = (3+t, 1+2t, 1-4t)$

vectors $(3,1,1)$ and $(1,2,-4)$

$\vec{r} = (3,1,1)$ ⟵ point on the line

$\vec{v} = (1,2,-4)$ ⟵ direction of the line

Important If you scale $\vec{v}$, that doesn't
change the line, it just changes the
parametrization

How to write as a soln to 2 eqns?
In each coord of:
$(x,y,z) = (3+t, 1+2t, 1-4t)$
There is a t!

→ we can solve:

$t = x-3$

$t = \frac{y-1}{2}$

$t = \frac{z-1}{-4} = \frac{1-z}{4}$

But all the same t.

⇒ if (x,y,z) is on the line, then:

$$x-3 = \frac{y-1}{2} = \frac{1-z}{4}$$

and conversely.

"symmetric form"

Caveat: If one of the coords of $\vec{v}$ is $\emptyset$, the
symmetric form looks a little different

$\vec{r} = (3,1,1)$ $\quad z = 1 + 0t = 1$
$\vec{v} = (1,2,0)$

$$\boxed{x-3 = \frac{y-1}{2} \quad \text{AND} \quad z = 1}$$ ⟵ symmetric form

What about writing a line through

What about writing a line through points P & Q?

Then set $\vec{v} = \overrightarrow{PQ}$

and $\vec{r} = P$ (as a vector)

or $\vec{r} = Q$ (as a vector)

eg. $P = (1, 1, 2)$ $\quad Q = (2, 0, 3)$

$\rightarrow$ Then $\vec{v} = (1, -1, 1)$

$\left.\begin{array}{l} \vec{r} = (1, 1, 2) \\ \text{or} \\ \vec{r} = (2, 0, 3) \end{array}\right\} \rightarrow$ get the same line

Similarly $\vec{v} = (-1, 1, -1)$ also gives the same line

(see Ex 1.19 in text)

Note: If L_1 is given by:

$$(x, y, z) = \vec{r}_1 + t_1 \vec{v}_1$$

and L_2 by:

$$(x, y, z) = \vec{r}_2 + t_2 \vec{v}_2$$

then $L_1 \parallel L_2$ iff $v_1 \parallel v_2$

and $L_1 \perp L_2$ iff $v_1 \perp v_2$

Note: In 2-space (Euclidean plane) the two lines either

① intersect

② are parallel

But in $\mathbb{R}^3$, they can also be skew ie, not parallel & don't intersect

example

Note, if $L_1 \perp L_2$ but don't intersect they are skew

([Co] 1.22)

Distance from a point to a line

Distance from a point to a line

Given point P and line L given by $\vec{r} + t\vec{v}$.

Then: The distance is

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$$

where $\vec{w}$ is the vector from $\vec{r}$ to P

eg. if $P = (2,3,1)$ & $\vec{r} = (3,0,-1)$

$\Rightarrow \vec{w} = (-1,3,2)$

Some symmetry

- If we scale $\vec{v}$ (replace $\vec{v}$ by $2\vec{v}$) then $\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$ doesn't change
- if we replace $\vec{w}$ by $-\vec{w}$ (eg take $\vec{w}$ to be from P to $\vec{r}$), then:

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$$

- If we choose a diff point $\vec{r}$ on the same line, the formula gives the same answer

Why? eg

replace $\vec{r}$ by $\vec{r} + 3\vec{v}$ then the effect $\vec{w}$ is to replace $\vec{w} = P - \vec{r}$

by $P - (\vec{r} + 3\vec{v})$

$= \vec{w} - 3\vec{v}$

then, if we replace $\vec{w}$ by $\vec{w} - 3\vec{v}$ in

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}, \text{ we get}$$

$$\frac{\|(\vec{w} - 3\vec{v}) \times \vec{v}\|}{\|\vec{v}\|}$$

$$\frac{\|(\vec{w}-3\vec{v})\times\vec{v}\|}{\|\vec{v}\|}$$

but

$$(\vec{w}-3\vec{v})\times\vec{v} = \vec{w}\times\vec{v} - 3\vec{v}\times\vec{v}$$

by bilinearity

$$= \vec{w}\times\vec{v} - \emptyset$$

(think 1.4 problem (27b))

Part 3: Planes

Recall: plane defined by 1 eqn

eg.

$ax+by+cz=d$ "normal form"

Notice can rewrite using dot prod.

$\vec{r}\cdot(a,b,c)=d$

with $\vec{r}=(x,y,z)$

Better way

1) Choose point on the plane $(x_0,y_0,z_0)=\vec{r}_0$

2) $\vec{r}_0\cdot(a,b,c)=d$

$\Rightarrow$ we can rewrite eqn as

$\vec{r}\cdot(a,b,c)=\vec{r}_0\cdot(a,b,c)$

equivalently:

$\vec{r}\cdot(a,b,c)-\vec{r}_0\cdot(a,b,c)=0$

3) Use bilinearity

$(\vec{r}-\vec{r}_0)\cdot(a,b,c)=0$

i.e., this eq. just says that $\vec{r}-\vec{r}_0 \perp (a,b,c)$

So $\vec{r}$ is in this plane iff $\vec{r}-r_0 \perp (a,b,c)$

"point-normal form"

How to get plane cont 3 points P, Q, R?

→ Can write parametrically as

$$(x, y, z) = \vec{P} + t\,\overrightarrow{PQ} + s\,\overrightarrow{PR}$$

Q: What if we want a normal vector?

ie — vector $\perp$ to $\overrightarrow{PQ}$ & $\overrightarrow{PR}$?

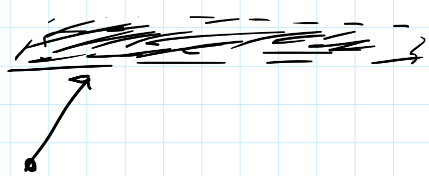
A: Cross-product

(Ex 1.2.4)

See formula 1.27 — distance from a point to a plane

→ can think of as $\dfrac{\| \vec{w} \cdot \vec{n} \|}{\| \vec{n} \|}$

for $\vec{n}$ normal vector and $\vec{w}$ from any point on plane to chosen point P.



— Read book for line of intersection of two planes.

Sect 1.2

1e) $\vec{v} = (-1, 5, -2)$

$\vec{w} = (3, 1, 1)$

Find $\| \frac{1}{2} (\vec{v} + \vec{w}) \|$

$\vec{v} + \vec{w} = (2, 6, -1)$

$\vec{v} + \vec{w} = (2, 6, -1)$

$\|\vec{v} + \vec{w}\| = \sqrt{2^2 + 6^2 + (-1)^2}$

$= \sqrt{4 + 36 + 1}$

$= \sqrt{41}$

$\frac{1}{2}\|\vec{v} + \vec{w}\| = \frac{\sqrt{41}}{2}$